Available online at www.sciencedirect.com

ScienceDirect

ELSEVIER

Linear Algebra and its Applications 422 (2007) 824–838

LINEAR ALGEBRA AND ITS APPLICATIONS

www.elsevier.com/locate/laa

Semiconvergence of two-stage iterative methods for singular linear systems ☆

Li Wang

School of Mathematics and Computer Sciences, Nanjing Normal University, Nanjing 210097, PR China

Received 2 March 2006; accepted 17 December 2006
Available online 23 December 2006
Submitted by H. Schneider

Abstract

We study the semiconvergence of two-stage iterative methods for solving nonsymmetric singular linear systems. The main tool we used to analyze their convergence is the so-called $\mathscr{R}(A)$-local P-regular splitting which is introduced in this paper.

AMS classification: 65F10

Keywords: Singular linear systems; Two-stage iterative method; Semiconvergence

1. Introduction

Consider a system of n equations

$$Ax = b, \tag{1.1}$$

where $A \in \mathscr{C}^{n\times n}$ is singular, $b, x \in \mathscr{C}^n$ with b known and x unknown. We assume that the linear system (1.1) is solvable, i.e., it has at least one solution. In order to solve the linear system (1.1) with iterative methods, the coefficient matrix A is often split into

$$A = M - N,$$

☆ Supported by National Natural Science Foundation No. 10371056, PR China, Founds for Major Basic Research Projects BK2006725, and College Natural Science Foundation No. 05KJB110062 of Jiangsu Province.

E-mail addresses: wangli1@njnu.edu.cn, wlsha@163.com

0024-3795/$ - see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.laa.2006.12.009

where M is a nonsingular matrix. Then a linear stationary iterative method for solving (1.1) can be described as follows:

$$x^{k+1} = Tx^k + M^{-1}b, \quad k = 0, 1, 2, \dots, \tag{1.2}$$

where $T = M^{-1}N$ is the iteration matrix.

It is well known that for the singular linear system (1.1) the iterative method (1.2) is semiconvergent if and only if the associated convergence factor

$$\gamma(T) \equiv \max\{|\mu| : \ \mu \in \sigma(T)\backslash\{1\}\} < 1$$

and the elementary divisors associated with $\mu = 1 \in \sigma(T)$ are linear, i.e.,

$$\operatorname{index}(I - T) = 1,$$

where $\sigma(T)$ denotes the spectrum of T and $\operatorname{index}(B)$ denotes the index of the matrix B, that is defined as the smallest nonnegative integer k such that $\operatorname{rank}(B^{k+1}) = \operatorname{rank}(B^k)$. In this case, T is called a semiconvergent matrix [7].

The iteration (1.2) would require the inversion of the matrix M. However, this is often costly and impractical in actual implementation. In this case, an approximate solution obtained with a few steps of the inner iteration induced by a splitting $M = F - G$ may be more attractive than an exact solution. Although this strategy can be expected to lead to some loss in the asymptotic rate of convergence of the outer iteration (compared to the one obtained if it were carried out exactly), the savings in computational work required to perform an outer iteration may be great enough to result in reduced solution times, see [3,4] for more details.

Two-stage iterative method, which is also called inner/outer iteration scheme, was first introduced by Nichols [17] and studied in depth by many authors, see [1,2,4,5,8–16]. But, in the past, attention has been directed almost exclusively to Hermitian systems or M-matrices. In this paper, we study the semiconvergence of two-stage iterative methods for solving the linear system (1.1), in the case that the coefficient matrix A is a nonsymmetric singular matrix.

Since the generalized inverse and the group inverse are important tools for singular linear system analysis, we recall their definitions [6] here as follows. For every matrix A, the generalized Moore–Penrose inverse $A^\dagger$ satisfies the following conditions:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger, \quad (A^\dagger A)^* = A^\dagger A,$$

and the group inverse $A^\#$ satisfies

$$AA^\# A = A, \quad A^\# AA^\# = A^\#, \quad A^\# A = AA^\#.$$

This paper is organized as follows. After reviewing some sufficient and necessary conditions which guarantee a splitting of a singular matrix to be convergent, in Sections 3 and 4, we investigate convergence conditions of two-stage iterative methods for singular linear systems. In Section 5, a few simple examples are given to illustrate the results presented in the previous sections.

2. Preliminary

For any matrix $B \in \mathscr{C}^{n\times n}$, we denote its Hermitian and skew-Hermitian parts as $\mathscr{H}(B) = \frac{1}{2}(B + B^*)$, $\mathscr{S}(B) = \frac{1}{2}(B - B^*)$, respectively, see [4,20]. $\mathscr{E}(T)$ represents a set which consists of the eigenvectors of T with at least one eigenvector associated with each distinct eigenvalue. In addition, several concepts which will be used in the sequel are defined as follows.

Definition 2.1

(i) A matrix $A \in \mathscr{C}^{n\times n}$ is Hermitian positive definite if it is Hermitian and for any $x \in \mathscr{C}^n\backslash\{0\}$, $x^*\mathscr{H}(A)x > 0$.

(ii) A matrix $A \in \mathscr{C}^{n\times n}$ is positive definite if for any $x \in \mathscr{C}^n\backslash\{0\}$, $x^*\mathscr{H}(A)x > 0$.

(iii) For a subset S of $\mathscr{C}^n$, a matrix $A \in \mathscr{C}^{n\times n}$ is S-positive definite if for any $x \in S\backslash\{0\}$, $x^*\mathscr{H}(A)x > 0$.

Definition 2.2 [18]. For $A \in \mathscr{C}^{n\times n}$, $A = M - N$ is called a P-regular splitting, if $M + N$ is positive definite.

Definition 2.3. For $A \in \mathscr{C}^{n\times n}$, $A = M - N$ is called a Hermitian convergent splitting if M and N are Hermitian matrices and $\rho(M^{-1}N) < 1$.

For convenience, we introduce the following concept.

Definition 2.4. For $A \in \mathscr{C}^{n\times n}$, $A = M - N$ is called S-local P-regular splitting if for any $x \in S\backslash\{0\}$ it holds that

$$\begin{cases} x^*Ax \neq 0, \\ x^*\mathscr{H}(2M - A)x \cdot x^*\mathscr{H}(A)x > x^*\mathscr{S}(2M - A)x \cdot x^*\mathscr{S}(A)x. \end{cases} \tag{2.1}$$

Lemma 2.1 [6]. *Let $A \in \mathscr{C}^{n\times n}$ be a singular matrix with $\mathscr{R}(A) = \mathscr{R}(A^*)$. Then $A^\#$ exists and $A^\# = A^\dagger$.*

Lemma 2.2 [18]. *Let $A \in \mathscr{C}^{n\times n}$ be Hermitian and $A = M - N$ be a P-regular splitting. Then $\rho(M^{-1}N) < 1$ if and only if A is Hermitian positive definite.*

Lemma 2.3. *Let $A = M - N$ where $A \in \mathscr{C}^{n\times n}$ is a singular matrix with $\mathscr{R}(A) = \mathscr{R}(A^*)$ and $x^*Mx \neq 0$ for $x \in \mathscr{E}(T) \cap \mathscr{R}(A)$, $T = M^{-1}N$. Then it holds that $x^*Ax \neq 0$ for any $x \in \mathscr{E}(T) \cap \mathscr{R}(A)$.*

Proof. If $x \in \mathscr{E}(T) \cap \mathscr{R}(A)$ and λ is the corresponding eigenvalue of T, then

$$Tx = \lambda x,$$

or equivalently,

$$Ax = (1 - \lambda)Mx. \tag{2.2}$$

Here, we note that $\mathscr{R}(A) = \mathscr{R}(A^*)$ and $\mathscr{R}(A^*) = \mathscr{N}(A)^\perp$ [6] where $\mathscr{N}(A)$ is the null space of the matrix A. So, if $x \in \mathscr{E}(T) \cap \mathscr{R}(A)$, we have $x \notin \mathscr{N}(A)$ and then $1 - \lambda \neq 0$.

Premultipling Eq. (2.2) with x^* goes the equality

$$x^*Ax = (1 - \lambda)x^*Mx.$$

Since for any $x \in \mathscr{E}(T) \cap \mathscr{R}(A)$, $x^*Mx \neq 0$ and $1 - \lambda \neq 0$, we can conclude that $x^*Ax \neq 0$ for any $x \in \mathscr{E}(T) \cap \mathscr{R}(A)$. □

Then, according to Theorem 2.2 in [19], the following result is direct.

Theorem 2.1. *Assume that a singular matrix $A \in \mathscr{C}^{n\times n}$ with $A = M - N$ and $T = M^{-1}N$ has the property $\mathscr{R}(A) = \mathscr{R}(A^*)$ and $x^*Mx \neq 0$ for any $x \in \mathscr{E}(T)$. Then, T is semiconvergent if and only if*

$$\frac{x^*(M^*(A^\dagger)^*A+N)x}{x^*Ax}>0 \tag{2.3}$$

$\forall x \in \mathscr{E}(T)\cap\mathscr{R}(A)$.

To guarantee a splitting of a singular matrix to be semiconvergent, we present a revised theorem which is a little different from that in [21]. For completeness, we also give its proof.

Theorem 2.2. *Let $A \in \mathscr{C}^{n\times n}$ be a singular matrix with $\mathscr{R}(A)=\mathscr{R}(A^*)$, $A = M - N$ and $T = M^{-1}N$. Assume that $x^*Mx \neq 0$ for any $x \in \mathscr{E}(T)$. Then, T is semiconvergent if and only if*

$$x^*\mathscr{H}(2M-A)x\cdot x^*\mathscr{H}(A)x > x^*\mathscr{S}(2M-A)x\cdot x^*\mathscr{S}(A)x \tag{2.4}$$

for any $x \in \mathscr{E}(T)\cap\mathscr{R}(A)$.

Proof. If $x \in \mathscr{E}(T)\cap\mathscr{R}(A)$ and λ is the corresponding eigenvalue of T, then

$$Tx=\lambda x,$$

which is equivalent to

$$Ax=(1-\lambda)Mx. \tag{2.5}$$

With the same reason given in the proof of Lemma 2.3, for $x \in \mathscr{E}(T)\cap\mathscr{R}(A)$, (2.5) implies that $x^*Ax \neq 0$, and by Theorem 2.1 we have $\gamma(T)<1$ if and only if

$$\frac{x^*(M^*(A^\dagger)^*A+N)x}{x^*Ax}>0.$$

Noting that $A^\dagger Ax=(1-\lambda)A^\dagger Mx$, we have

$$\begin{aligned}
\frac{x^*(M^*(A^\dagger)^*A+N)x}{x^*Ax} &= \frac{\frac{x^*(A^\dagger A)^*Ax}{1-\bar\lambda}+x^*Nx}{x^*Ax}\\
&= \frac{\frac{x^*AA^\dagger Ax}{1-\bar\lambda}+x^*Nx}{x^*Ax}\\
&= \frac{\frac{x^*Ax}{1-\bar\lambda}+x^*Nx}{x^*Ax}\\
&= \frac{\left(\frac{x^*Ax}{1-\frac{x^*N^*x}{x^*M^*x}}+x^*Nx\right)x^*A^*x}{x^*Ax\cdot x^*A^*x}\\
&= \frac{x^*Ax\cdot x^*M^*x+x^*Nx\cdot x^*A^*x}{x^*Ax\cdot x^*A^*x}\\
&= \frac{x^*Ax\cdot x^*M^*x+x^*Mx\cdot x^*A^*x-x^*Ax\cdot x^*A^*x}{x^*Ax\cdot x^*A^*x}\\
&> 0,
\end{aligned}$$

which is equivalent to (2.4).

Now, by Theorem 2.1, the result follows immediately. □

3. Semiconvergence of stationary two-stage iteration

For $A \in \mathscr{C}^{n\times n}$, we let $A = M - N$ and $M = F - G$. Then the stationary two-stage iteration algorithm can be formulated as follows [17].

Algorithm 3.1. (*Stationary two-stage iteration*)
Given an initial vector x_0
For $k = 1, 2, \ldots,$
set $y_0 = x_{k-1}$,
For $j = 1, \ldots, p$,
solve $Fy_j = Gy_{j-1} + Nx_{k-1} + b$.
Set $x_k = y_p$,
where the positive integer p is the number of inner iterations.

From Algorithm 3.1, we have

$$\begin{aligned} x_k &= (F^{-1}G)^p x_{k-1} + \sum_{j=0}^{p-1} (F^{-1}G)^j F^{-1}(Nx_{k-1} + b) \\ &= H^p x_{k-1} + (I - H^p)(I - H)^{-1} F^{-1}(Nx_{k-1} + b), \end{aligned} \tag{3.1}$$

where $H = F^{-1}G$ and $I - H$ is assumed to be nonsingular. If we further assume that $I - H^p$ is also nonsingular, then the total iterative matrix of Algorithm 3.1 is

$$T_p = H^p + (I - H^p)(I - H)^{-1}F^{-1}N, \tag{3.2}$$

which can be induced by the splitting $A = M_{T_p} - N_{T_p}$, where

$$M_{T_p} = F(I - H)(I - H^p)^{-1}, \quad N_{T_p} = M_{T_p} - A. \tag{3.3}$$

Theorem 3.1. *Let $A \in \mathscr{C}^{n\times n}$ be an $\mathscr{R}(A)$-positive definite singular matrix and satisfy $\mathscr{R}(A) = \mathscr{R}(A^*)$, $A = M - N$ an $\mathscr{R}(A)$-local P-regular splitting, M a Hermitian positive definite matrix, $M = F - G$ a Hermitian convergent splitting and $H = F^{-1}G$. Then $I - H^p$ is nonsingular and the two-stage Algorithm 3.1 is semiconvergent, if the inner iteration number p is even. Moreover, $A = M_{T_p} - N_{T_p}$ is an $\mathscr{R}(A)$-local P-regular splitting.*

Proof. Since $M = F - G$ is a Hermitian convergent splitting, $\rho(H) \equiv \rho(F^{-1}G) < 1$. Thus $I - H^p$ is nonsingular, and we can rewrite M_{T_p} as

$$\begin{aligned} M_{T_p} &= F(I - F^{-1}G)(I - (F^{-1}G)^p)^{-1} \\ &= M(I - H^p)^{-1}. \end{aligned} \tag{3.4}$$

By using (3.3) and (3.4), we have

$$\begin{aligned} M_{T_p} + N_{T_p} &= 2M(I - H^p)^{-1} - A \\ &= 2M(I - H^p)^{-1} - 2M + M + N \end{aligned}$$

$$
\begin{aligned}
&= 2M \sum_{j=0}^{\infty} H^{pj} H^{p} + M + N \\
&= 2M \sum_{j=1}^{\infty} H^{pj} + M + N.
\end{aligned} \tag{3.5}
$$

In addition, for any even positive integer p and $j = 1, 2, \ldots,$ we have

$$
\begin{aligned}
MH^{pj} &= (F - G)(F^{-1}G)^{pj} \\
&= G(F^{-1}G)^{pj-1} - G(F^{-1}G)^{pj} \\
&= G(F^{-1}G)^{\frac{pj}{2}-1} F^{-1}G(F^{-1}G)^{\frac{pj}{2}-1} \\
&\quad - G(F^{-1}G)^{\frac{pj}{2}-1} F^{-1}GF^{-1}G(F^{-1}G)^{\frac{pj}{2}-1} \\
&= G(F^{-1}G)^{\frac{pj}{2}-1}(F^{-1} - F^{-1}GF^{-1})G(F^{-1}G)^{\frac{pj}{2}-1} \\
&= (GF^{-1})^{\frac{pj}{2}} M(F^{-1}G)^{\frac{pj}{2}}.
\end{aligned} \tag{3.6}
$$

Since M is a Hermitian positive definite matrix and $M = F - G$ is a Hermitian convergent splitting, the last equality of (3.6) implies that for any positive integer j, MH^{pj} is Hermitian positive semidefinite, and M_{T_p}, i.e. $M\sum_{j=0}^{\infty} H^{pj}$, is Hermitian positive definite. So, with (3.5), we obtain

$$
\begin{aligned}
\mathscr{S}(2M_{T_p} - A) &= \mathscr{S}(M_{T_p} + N_{T_p}) \\
&= \mathscr{S}\left(2M \sum_{j=1}^{\infty} H^{pj} + M + N\right) \\
&= \mathscr{S}(M + N) \\
&= \mathscr{S}(2M - A).
\end{aligned} \tag{3.7}
$$

Because A is an $\mathscr{R}(A)$-positive definite matrix and $A = M - N$ is an $\mathscr{R}(A)$-local P-regular splitting, then, with (2.1), for any $x \in \mathscr{R}(A)\backslash\{0\}$, we have

$$
\begin{cases}
x^*Ax \neq 0, \\
x^*\mathscr{H}(2M - A)x \cdot x^*\mathscr{H}(A)x > x^*\mathscr{S}(2M - A)x \cdot x^*\mathscr{S}(A)x.
\end{cases}
$$

Hence, by (3.5) and (3.7), we have

$$
\begin{aligned}
&x^*\mathscr{H}(2M_{T_p} - A)xx^*\mathscr{H}(A)x \\
&\quad = x^*\mathscr{H}(M_{T_p} + N_{T_p})xx^*\mathscr{H}(A)x \\
&\quad = x^*\left(\mathscr{H}(M + N) + 2M \sum_{j=1}^{\infty} H^{pj}\right) xx^*\mathscr{H}(A)x \\
&\quad = x^*\left(\mathscr{H}(2M - A) + 2M \sum_{j=1}^{\infty} H^{pj}\right) xx^*\mathscr{H}(A)x
\end{aligned}
$$

$$
\begin{aligned}
&= x^*\mathscr{H}(2M-A)xx^*\mathscr{H}(A)x + x^*\left(2M\sum_{j=1}^{\infty}H^{pj}\right)xx^*\mathscr{H}(A)x\\
&> x^*\mathscr{S}(2M-A)xx^*\mathscr{S}(A)x + x^*\left(2M\sum_{j=1}^{\infty}H^{pj}\right)xx^*\mathscr{H}(A)x\\
&\geqslant x^*\mathscr{S}(2M-A)xx^*\mathscr{S}(A)x\\
&= x^*\mathscr{S}(2M_{T_p}-A)xx^*\mathscr{S}(A)x,
\end{aligned}\tag{3.8}
$$

which means that $A = M_{T_p} - N_{T_p}$ is an $\mathscr{R}(A)$-local P-regular splitting.

Noting that M_{T_p} is Hermitian positive definite and $\mathscr{R}(A)\cap\mathscr{E}(T_p)\subset\mathscr{R}(A)\backslash\{0\}$, we have $x^*M_{T_p}x \neq 0$ for $x\in\mathscr{E}(T_p)$ and

$$x^*\mathscr{H}(2M_{T_p}-A)x\cdot x^*\mathscr{H}(A)x > x^*\mathscr{S}(2M_{T_p}-A)x\cdot x^*\mathscr{S}(A)x$$

for $x\in\mathscr{R}(A)\cap\mathscr{E}(T_p)$. By Theorem 2.2, the results are obtained. □

In the following we will give convergence results in which the inner iteration number p can be allowed to be any positive integer.

Theorem 3.2. *Let $A\in\mathscr{C}^{n\times n}$ be an $\mathscr{R}(A)$-positive definite singular matrix and satisfy $\mathscr{R}(A)=\mathscr{R}(A^*)$, $A = M - N$ an $\mathscr{R}(A)$-local P-regular splitting, M a Hermitian positive definite matrix, $M = F - G$ a Hermitian splitting with G Hermitian positive semidefinite and $H = F^{-1}G$. Then $I - H^p$ is nonsingular and the two-stage Algorithm 3.1 is semiconvergent, if the inner iteration number p is any positive integer. Moreover, $A = M_{T_p} - N_{T_p}$ is an $\mathscr{R}(A)$-local P-regular splitting.*

Proof. Since M is Hermitian positive definite and G is Hermitian positive semidefinite, then $F = M + G$ and $F + G = M + 2G$ are Hermitian positive definite. Thus, $M = F - G$ is a Hermitian P-regular splitting, Lemma 2.2 implies $\rho(H)\equiv\rho(F^{-1}G) < 1$, and hence $I - H^p$ is nonsingular.

Thus, we can rewrite M_{T_p} and $M_{T_p} + N_{T_p}$ as (3.4) and (3.5), respectively. We now consider the terms MH^{pj}, $j = 1, 2, \dots,$ in (3.5).

If pj is even, then (cf. (3.6)), MH^{pj} is Hermitian positive semidefinite.

If pj is odd, then

$$
\begin{aligned}
MH^{pj} &= (F-G)(F^{-1}G)^{pj}\\
&= G(F^{-1}G)^{pj-1} - G(F^{-1}G)^{pj}\\
&= (GF^{-1})^{\frac{pj-1}{2}}G(F^{-1}G)^{\frac{pj-1}{2}} - (GF^{-1})^{\frac{pj-1}{2}}GF^{-1}G(F^{-1}G)^{\frac{pj-1}{2}}\\
&= (GF^{-1})^{\frac{pj-1}{2}}(G - GF^{-1}G)(F^{-1}G)^{\frac{pj-1}{2}}.
\end{aligned}\tag{3.9}
$$

In addition, we know that

$$G - GF^{-1}G = G^{\frac{1}{2}}(I - G^{\frac{1}{2}}F^{-1}G^{\frac{1}{2}})G^{\frac{1}{2}}\tag{3.10}$$

is Hermitian positive semidefinite, since $G^{\frac{1}{2}}F^{-1}G^{\frac{1}{2}}$ is Hermitian positive semidefinite and $\rho(G^{\frac{1}{2}}F^{-1}G^{\frac{1}{2}}) = \rho(F^{-1}G) < 1$.

Now, we can deduce that, for any positive integers j and p, MH^{pj} is Hermitian positive semidefinite, and M_{T_p}, i.e., $M\sum_{j=0}^{\infty} H^{pj}$, is Hermitian positive definite. So, with (3.5), (3.7) holds true.

Since A is $\mathscr{R}(A)$-positive definite and $A = M - N$ is an $\mathscr{R}(A)$-local P-regular splitting, by (3.8), $A = M_{T_p} - N_{T_p}$ is an $\mathscr{R}(A)$-local P-regular splitting, too.

Noting that M_{T_p} is Hermitian positive definite and $\mathscr{R}(A) \cap \mathscr{E}(T_p) \subset \mathscr{R}(A)\backslash\{0\}$, we have $x^*M_{T_p}x \neq 0$ for $x \in \mathscr{E}(T_p)$ and

$$x^*\mathscr{H}(2M_{T_p} - A)x \cdot x^*\mathscr{H}(A)x > x^*\mathscr{S}(2M_{T_p} - A)x \cdot x^*\mathscr{S}(A)x$$

for $x \in \mathscr{R}(A) \cap \mathscr{E}(T_p)$. By Theorem 2.2, the results are obtained. □

4. Semiconvergence of block two-stage iteration

We partition the singular matrix $A \in \mathscr{C}^{n\times n}$ in (1.1) into $q \times q$ blocks

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1q} \\ A_{21} & A_{22} & \cdots & A_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ A_{q1} & A_{q2} & \cdots & A_{qq} \end{pmatrix} \tag{4.1}$$

with the diagonal blocks $A_{ii} \in \mathscr{C}^{n_i\times n_i}$, $i = 1, \ldots, q$, and $\sum_{i=1}^{q} n_i = n$.

In the splitting $A = M - N$, M is block diagonal, denoted by $M = \mathrm{diag}(M_i)$, with the blocks $M_i \in \mathscr{C}^{n_i\times n_i}$ being nonsingular but not necessarily equal to A_{ii}, $i = 1, \ldots, q$.The vectors x, b and other intermediate vectors are partitioned in the way consistent with (4.1). If splittings $M_i = F_i - G_i$, $i = 1, \ldots, q$, are used, the block two-stage iterative method is the following [11].

Algorithm 4.1 (*Block two-stage iteration*). Given an initial vector $x_0 = \left[x_0^{(1)\mathrm{T}}, \ldots, x_0^{(q)\mathrm{T}}\right]^{\mathrm{T}}$.

For $k = 1, 2, \ldots,$

For $i = 1, \ldots, q$,

set $y_0^{(i)} = x_{k-1}^{(i)}$.

For $j = 1, \ldots, p_{k,i}$,

solve $F_i y_j^{(i)} = G_i y_{j-1}^{(i)} + (Nx_{k-1} + b)^{(i)}$.

Set $x_k^{(i)} = y_{p_{k,i}}^{(i)}$,

where positive integers $p_{k,i}$, $k = 1, 2, \ldots,$ $i = 1, \ldots, q$, are numbers of inner iterations, which may depend on k and i.

If $q = 1$ and $p_{k,i} = p$ for all k and i, then Algorithm 4.1 is reduced to stationary two-stage iterative method. In this section, we consider the convergence of Algorithm 4.1, where we let $p_{k,i} = p_i$ for any k. From this algorithm we have

$$x_k^{(i)} = (F_i^{-1}G_i)^{p_i} x_{k-1}^{(i)} + \sum_{j=0}^{p_i-1} (F_i^{-1}G_i)^j F_i^{-1}(Nx_{k-1} + b)^{(i)}, \quad i = 1, \ldots, q, \tag{4.2}$$

where $A = M - N$, $M = \mathrm{diag}(M_i)$, $M_i = F_i - G_i$, $i = 1, \ldots, q$.

Then

$$x_k = \left(x_k^{(1)\mathrm{T}}, x_k^{(2)\mathrm{T}}, \ldots, x_k^{(q)\mathrm{T}}\right)^{\mathrm{T}}$$

$$= \mathrm{diag}((F_i^{-1}G_i)^{p_i})x_{k-1} + \mathrm{diag}\left(\sum_{j=0}^{p_i-1}(F_i^{-1}G_i)^j F_i^{-1}\right)(Nx_{k-1}+b)$$
$$= \mathrm{diag}({H_i}^{p_i})x_{k-1} + \mathrm{diag}\left(\sum_{j=0}^{p_i-1}{H_i}^j F_i^{-1}\right)(Nx_{k-1}+b)$$
$$= \mathrm{diag}({H_i}^{p_i})x_{k-1} + \mathrm{diag}((I_{n_i}-H_i^{p_i})(I_{n_i}-H_i)^{-1}F_i^{-1})(Nx_{k-1}+b), \tag{4.3}$$

where $H_i = F_i^{-1}G_i$, I_{n_i} is $n_i \times n_i$ identity matrix, $(\cdot)^{\mathrm{T}}$ is the transpose of a vector $(\cdot)$, and we have assumed that $I_{n_i} - H_i$ is nonsingular. If we assume that $I_{n_i} - H_i^{p_i}$ is also nonsingular, the iterative formula of Algorithm 4.1 can be written as

$$x_k = T_p x_{k-1} + M_{T_p}^{-1} b, \tag{4.4}$$

where the total iterative matrix of Algorithm 4.1 is

$$T_p = \mathrm{diag}(H_i^{p_i}) + \mathrm{diag}((I_{n_i}-H_i^{p_i})(I_{n_i}-H_i)^{-1}F_i^{-1})N, \tag{4.5}$$

which can be induced by the splitting $A = M_{T_p} - N_{T_p}$, with

$$M_{T_p} = \mathrm{diag}(F_i(I_{n_i}-H_i)(I_{n_i}-H_i^{p_i})^{-1}), \quad N_{T_p} = M_{T_p} - A. \tag{4.6}$$

Theorem 4.1. *Let $A \in \mathscr{C}^{n\times n}$ be an $\mathscr{R}(A)$-positive definite singular matrix and satisfy $\mathscr{R}(A) = \mathscr{R}(A^*)$, $A = M - N$ an $\mathscr{R}(A)$-local P-regular splitting, M a Hermitian positive definite matrix, $M = F - G$ a Hermitian convergent splitting and $H = F^{-1}G$. Here, A is partitioned into $q \times q$ blocks as in (4.1), $M = \mathrm{diag}(M_i)$, $M_i \in \mathscr{C}^{n_i\times n_i}$, and $M_i = F_i - G_i$, $i = 1, 2, \ldots, q$, are Hermitian convergent splittings. Then $I_{n_i} - H_i^{p_i}$, $i = 1, 2, \ldots, q$, are nonsingular and the two-stage Algorithm 4.1 is semiconvergent, if p_i, $i = 1, 2, \ldots, q$, are even positive integers. Moreover, $A = M_{T_p} - N_{T_p}$ is an $\mathscr{R}(A)$-local P-regular splitting.*

Proof. Since $M = F - G$ is a Hermitian convergent splitting, $\rho(H) \equiv \rho(F^{-1}G) < 1$ and $\rho(H_i) \equiv \rho(F_i^{-1}G_i) < 1, i = 1, 2, \ldots, q$. Thus, $I_{n_i} - H_i^{p_i}$, $i = 1, 2, \ldots, q$, and $\mathrm{diag}(I_{n_i} - H_i^{p_i})$ are nonsingular.

From this point on, the proof copies that of Theorem 3.1 except that the matrices M, F, G, H are replaced by the matrices $\mathrm{diag}(M_i)$, $\mathrm{diag}(F_i)$, $\mathrm{diag}(G_i)$, $\mathrm{diag}(H_i)$, respectively. For example, instead of (3.4),

$$M_{T_p} = F(I - F^{-1}G)(I - (F^{-1}G)^p)^{-1}$$
$$= M(I - H^p)^{-1},$$

we have now

$$M_{T_p} = \mathrm{diag}(F_i(I_{n_i}-H_i)(I_{n_i}-H_i^{p_i})^{-1})$$
$$= \mathrm{diag}(M_i(I_{n_i}-H_i^{p_i})^{-1}), \tag{4.7}$$

and, to guarantee $\mathrm{diag}\big(M_i \sum_{j=0}^{\infty} H_i^{p_i j}\big)$ to be Hermitian positive definite, for any even positive integer p_i, $i = 1, 2, \ldots, q$, and any positive index, we rewrite $M_i H_i^{p_i j}$ in the similar form as in (3.6), etc., then, we can obtain our results. $\square$

In the following, we will further give convergence results for which the inner iteration numbers p_i, $i = 1, 2, \ldots, q$, are allowed to be any positive integers.

Theorem 4.2. *Let $A \in \mathscr{C}^{n\times n}$ be an $\mathscr{R}(A)$-positive definite singular matrix and satisfy $\mathscr{R}(A) = \mathscr{R}(A^*)$, $A = M - N$ an $\mathscr{R}(A)$-local P-regular splitting, M a Hermitian positive definite matrix, $M = F - G$ a Hermitian convergent splitting with G Hermitian positive semidefinite, and $H = F^{-1}G$. Here, A is partitioned into $q \times q$ blocks as in (4.1), $M = \operatorname{diag}(M_i)$, $M_i \in \mathscr{C}^{n_i\times n_i}$, and $M_i = F_i - G_i$, $i = 1, 2, \ldots, q$, are such that G_i, $i = 1, 2, \ldots, q$, are Hermitian positive semi-definite. Then $I_{n_i} - H_i^{p_i}$, $i = 1, 2, \ldots, q$, are nonsingular and the two-stage Algorithm 4.1 is semiconvergent, if p_i, $i = 1, 2, \ldots, q$, are any positive integers. Moreover, $A = M_{T_p} - N_{T_p}$ is an $\mathscr{R}(A)$-local P-regular splitting.*

Proof. Since M is block diagonal and Hermitian positive definite, $M = \operatorname{diag}(M_i)$, and G_i, $i = 1, 2, \ldots, q$, are Hermitian positive semidefinite, then $F_i = M_i + G_i$, $i = 1, 2, \ldots, q$, and $F_i + G_i = M_i + 2G_i$, $i = 1, 2, \ldots, q$, are Hermitian positive definite. Thus, $M_i = F_i - G_i$, $i = 1, 2, \ldots, q$, are Hermitian P-regular splittings. Lemma 2.2 implies that $\rho(H_i) \equiv \rho(F_i^{-1}G_i) < 1$, $i = 1, 2, \ldots, q$, and, hence, $I_{n_i} - H_i^{p_i}$, $i = 1, 2, \ldots, q$ and $\operatorname{diag}(I_{n_i} - \operatorname{diag}(H_i^{p_i}))$ are nonsingular.

Now, similar to Theorem 4.1, exactly following the demonstration of the proof of Theorem 3.2 except that the matrices M, F, G, H are replaced by the matrices $\operatorname{diag}(M_i)$, $\operatorname{diag}(F_i)$, $\operatorname{diag}(G_i)$, $\operatorname{diag}(H_i)$, respectively, we can complete the proof. $\square$

5. Numerical examples

First, we give the following results which will be useful in the sequel discussions.

Lemma 5.1. *Assume $A \in \mathscr{C}_r^{n\times n}$ and $r < n$, i.e., A is an $n \times n$ complex matrix with rank r. Then $\mathscr{R}(A) = \mathscr{R}(A^*)$ if and only if there exist a matrix $E \in \mathscr{C}_r^{n\times r}$ and a matrix $B \in \mathscr{C}_r^{r\times r}$ such that $A = EBE^*$.*

Proof. According to the full rank decomposition theorem [6], there exist $E \in \mathscr{C}_r^{n\times r}$ and $F \in \mathscr{C}_r^{r\times n}$ such that $A = EF$. Then, $\mathscr{R}(A) = \mathscr{R}(E)$, since $\mathscr{R}(A) \subseteq \mathscr{R}(E)$ and $\operatorname{rank}(A) = \operatorname{rank}(E) = r$. Noting that $A^* = F^*E^*$, we have $\mathscr{R}(A^*) = \mathscr{R}(F^*)$.

If $\mathscr{R}(A) = \mathscr{R}(A^*)$, from the above analysis we can obtain $\mathscr{R}(E) = \mathscr{R}(F^*)$. Therefore, there exists a matrix $B \in \mathscr{C}_r^{r\times r}$ such that $F^* = EB^*$. The necessity is proved.

Conversely, if $A = EBE^*$, where $E \in \mathscr{C}_r^{n\times r}$ and $B \in \mathscr{C}_r^{r\times r}$, then $\mathscr{R}(A) = \mathscr{R}(EB) = \mathscr{R}(EB^*) = \mathscr{R}(A^*) = \mathscr{R}(E)$. The proof is completed. $\square$

Theorem 5.1. *Let $A \in \mathscr{C}_r^{n\times n}$, $r < n$, be an $\mathscr{R}(A)$-positive definite singular matrix and satisfy $\mathscr{R}(A) = \mathscr{R}(A^*)$. Then there exist matrices $E \in \mathscr{C}_r^{n\times r}$ and $B \in \mathscr{C}_r^{r\times r}$ such that $A = EBE^*$ and a splitting $A = M - N$ is an $\mathscr{R}(A)$-local P-regular splitting of A, if M is Hermitian positive definite and*

$$\max\left\{\frac{\rho(E^*\mathscr{H}(A)E)^2 + \rho(E^*\mathscr{S}(A)E)^2}{\rho(E^*\mathscr{H}(A)E)\nu(E^*\mathscr{H}(M)E)}, \frac{\nu(E^*\mathscr{H}(A)E)^2 + \rho(E^*\mathscr{S}(A)E)^2}{\nu(E^*\mathscr{H}(A)E)\nu(E^*\mathscr{H}(M)E)}\right\} < 2, \tag{5.1}$$

where $\nu(C)$ represents the minimal absolute value of the eigenvalues of the corresponding matrix C.

Proof. By Lemma 5.1, there exist matrices $E \in \mathscr{C}_r^{n\times r}$ and $B \in \mathscr{C}_r^{r\times r}$ such that $A = EBE^*$. It is not difficult to see that $\mathscr{R}(A)\backslash\{0\} = \{Ex : x \in \mathscr{C}^r\backslash\{0\}\}$.

Since A is an $\mathscr{R}(A)$-positive definite matrix and M is a positive definite matrix, by (5.1) we have

$$\begin{aligned}
&\max_{x\in\mathscr{R}(A)\backslash\{0\}}\left\{\frac{(x^*\mathscr{H}(A)x)^2-(x^*\mathscr{S}(A)x)^2}{x^*\mathscr{H}(A)x\cdot x^*\mathscr{H}(M)x}\right\}\\
&\quad=\max_{x\in\mathscr{C}^r\backslash\{0\}}\left\{\frac{(x^*E^*\mathscr{H}(A)Ex)^2-(x^*E^*\mathscr{S}(A)Ex)^2}{x^*E^*\mathscr{H}(A)Ex\cdot x^*E^*\mathscr{H}(M)Ex}\right\}\\
&\quad\leqslant\max_{x\in\mathscr{C}^r,x^*x=1}\left\{\frac{(x^*E^*\mathscr{H}(A)Ex)^2+\rho(E^*\mathscr{S}(A)E)^2}{x^*E^*\mathscr{H}(A)Ex\nu(E^*\mathscr{H}(M)E)}\right\}\\
&\quad\leqslant\max\left\{\frac{\rho(E^*\mathscr{H}(A)E)^2+\rho(E^*\mathscr{S}(A)E)^2}{\rho(E^*\mathscr{H}(A)E)\nu(E^*\mathscr{H}(M)E)},\frac{\nu(E^*\mathscr{H}(A)E)^2+\rho(E^*\mathscr{S}(A)E)^2}{\nu(E^*\mathscr{H}(A)E)\nu(E^*\mathscr{H}(M)E)}\right\}\\
&\quad<2,
\end{aligned}$$

which means, for any $x \in \mathscr{R}(A)\backslash\{0\}$,

$$2x^*\mathscr{H}(M)x\cdot x^*\mathscr{H}(A)x > (x^*\mathscr{H}(A)x)^2-(x^*S(A)x)^2.$$

Note that the last inequality is nothing but the one in (2.1) when the matrix A is positive definite and the matrix M is Hermitian positive definite.

So, $A = M - N$ is an $\mathscr{R}(A)$-local P-regular splitting. $\square$

In the following, we use two examples to further illustrate the conditions and examine the correctness of the theorems in Sections 3 and 4.

Example 5.1. Let

$$A=\begin{bmatrix}0.4&0.96&0&0&1.36\\-1.04&0.4&1.96&0&1.32\\0&-2.04&0.4&0.96&-0.68\\0&0&-1.04&0.4&-0.64\\-0.64&-0.68&1.32&1.36&1.36\end{bmatrix},$$

which can be decomposed as

$$A = EBE^*,$$

where

$$E=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\1&1&1&1\end{bmatrix},\quad B=\begin{bmatrix}0.4&0.96&0&0\\-1.04&0.4&1.96&0\\0&-2.04&0.4&0.96\\0&0&-1.04&0.4\end{bmatrix}.$$

It is easy to know that $\mathscr{R}(A) = \mathscr{R}(A^*) = \mathscr{R}(E)$ and A is $\mathscr{R}(A)$-positive definite. If we choose

$$M = \begin{bmatrix} 424.4 & -0.04 & 0 & 0 & 0.36 \\ -0.04 & 424.4 & -0.04 & 0 & 0.32 \\ 0 & -0.04 & 424.4 & -0.04 & 0.32 \\ 0 & 0 & -0.04 & 424.4 & 0.36 \\ 0.36 & 0.32 & 0.32 & 0.36 & 425.36 \end{bmatrix},$$

$$F = \begin{bmatrix} 323.6 & 76.8 & 0 & 0 & 0 \\ 76.8 & 400.4 & 76.8 & 0 & 0 \\ 0 & 76.8 & 323.6 & 76.8 & 0 \\ 0 & 0 & 76.8 & 400.4 & 76.8 \\ 0 & 0 & 0 & 76.8 & 323.6 \end{bmatrix},$$

then $N = M - A$, $G = F - M$,

$$H = \begin{bmatrix} -0.3774 & 0.2776 & -0.0692 & 0.0139 & -0.0043 \\ 0.2776 & -0.1691 & 0.2915 & -0.0586 & 0.0135 \\ -0.0692 & 0.2914 & -0.4499 & 0.2914 & -0.07 \\ 0.0141 & -0.0584 & 0.2917 & -0.1688 & 0.2773 \\ -0.0045 & 0.0129 & -0.0702 & 0.2763 & -0.3803 \end{bmatrix},$$

and obviously, M is Hermitian positive definite. Noting that $\rho(E^*S(A)E) = 5.3983$, $\rho(E^*\mathscr{H}(A)E) = 8.5012$, $\nu(E^*\mathscr{H}(A)E) = 0.3753$ and $\nu(E^*ME) = 424.4$, similar to Example 5.2, we can see that

$$\begin{aligned} &\max\left\{\frac{\rho(E^*\mathscr{H}(A)E)^2 + \rho(E^*\mathscr{S}(A)E)^2}{\rho(E^*\mathscr{H}(A)E)\nu(E^*\mathscr{H}(M)E)}, \frac{\nu(E^*\mathscr{H}(A)E)^2 + \rho(E^*\mathscr{S}(A)E)^2}{\nu(E^*\mathscr{H}(A)E)\nu(E^*\mathscr{H}(M)E)}\right\} \\ &= 0.1838 \\ &< 2. \end{aligned}$$

So, $A = M - N$ is an $\mathscr{R}(A)$-local P-regular splitting, and $M = F - G$ is a Hermitian convergent splitting. By Theorem 3.1, we see that, for any even positive integer p, the two-stage iterative method should be semiconvergent.

In fact, some computations yield that

$$\text{for } p = 2, 4, 6, 8, \quad \gamma(T_p) = 0.9993, 0.9986, 0.9985, 0.9984,$$

respectively. This further illustrate the correctness of Theorem 3.1.

Example 5.2. Let

$$A = \begin{bmatrix} 3.6 & 3.25 & 5.55 & 4.55 \\ 1.25 & 3.6 & 4.9 & 3.55 \\ 2.55 & 5.9 & 8.55 & 5.8 \\ 3.55 & 4.55 & 6.8 & 5.85 \end{bmatrix},$$

which can be decomposed as

$$A = EBE^*,$$

where

$$E=\begin{bmatrix}1&1&0\\0&1&1\\0&2&1\\1&1&1\end{bmatrix},\quad B=\begin{bmatrix}1.35&0.95&0\\-0.05&1.35&0.95\\0&-0.05&1.35\end{bmatrix}.$$

It is easy to see that $\mathscr{R}(A)=\mathscr{R}(A^*)=\mathscr{R}(E)$ and A is $\mathscr{R}(A)$-positive definite. If we choose

$$M=\begin{bmatrix}210.42&-105.21&0&0\\-105.21&210.42&-105.21&0\\0&-105.21&210.42&-105.21\\0&0&-105.21&210.42\end{bmatrix},$$

$$F=\begin{bmatrix}400&0&0&0\\0&400&0&0\\0&0&400&0\\0&0&0&400\end{bmatrix},$$

then $N=M-A$, $G=F-M$,

$$H=\begin{bmatrix}0.47395&0.263025&0&0\\0.263025&0.47395&0.263025&0\\0&0.263025&0.47395&0.263025\\0&0&0.263025&0.47395\end{bmatrix},$$

and obviously, M is Hermitian positive definite.

By computation, we know that $\rho(E^*\mathscr{S}(A)E)=7.433$, $\rho(E^*\mathscr{H}(A)E)=195.219$, $\nu(E^*\mathscr{H}(A)E)=0.185$, and $\nu(E^*\mathscr{H}(M)E)=159.794$.

We also have

$$\begin{aligned}&\max\left\{\frac{\rho(E^*\mathscr{H}(A)E)^2+\rho(E^*\mathscr{S}(A)E)^2}{\rho(E^*\mathscr{H}(A)E)\nu(E^*\mathscr{H}(M)E)},\frac{\nu(E^*\mathscr{H}(A)E)^2+\rho(E^*\mathscr{S}(A)E)^2}{\nu(E^*\mathscr{H}(A)E)\nu(E^*\mathscr{H}(M)E)}\right\}\\&=1.8701\\&<2.\end{aligned}$$

So, by Theorem 5.1, $A=M-N$ is an $\mathscr{R}(A)$-local P-regular splitting. In addition, $M=F-G$ is a Hermitian convergent splitting and G is a Hermitian positive definite matrix. By Theorem 3.2, we see that, for any positive integer p, the two-stage iterative method should be semiconvergent.

In fact, some computations yield that for $p=1,2,3,4,5,6$,

$$\gamma(T_p)=0.998796, 0.998635, 0.998564, 0998521, 0.998495, 0.998478,$$

respectively. This further illustrate the correctness of Theorem 3.2.

If $\rho(H)<1$, by (3.2), it holds that $T_p\to T=M^{-1}N$ as $p\to\infty$. The above two examples show that in order to obtain a better asymptotic convergence rate, we should increase the inner iteration number p, but this will increase the workload. In either of the two examples, the inverse of F is easier to construct than that of M. So, as mentioned in Section 1, the two-stage iterative methods can reduce computational work and storage, but may sacrifice the convergence speed a little.

Remark 5.1. If $\mathscr{R}(A)=\mathscr{R}(A^*)$, then $\operatorname{index}(A)=1$, but if $\operatorname{index}(A)=1$, it is not necessary for $\mathscr{R}(A)=\mathscr{R}(A^*)$ being hold. An example is given as follows:

$$A=\begin{bmatrix}1&0&1\\0&-1&1\\1&-1&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&-1&1\\1&-1&0\end{bmatrix}^{-1}.$$

Obviously, $\operatorname{index}(A)=1$. The null space of A is

$$\mathscr{N}(A)=\left\{\begin{bmatrix}1&0&1\\0&-1&1\\1&-1&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\0\end{bmatrix},\ y_1,y_2\in\mathscr{C}\right\}.$$

Since $\mathscr{R}(A^*)=\mathscr{N}(A)^{\perp}$ [6] and for $y_1\neq y_2$, $y_1,y_2\in\mathscr{C}$,

$$\begin{aligned}&\left(\begin{bmatrix}1&0&1\\0&-1&1\\1&-1&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\0\end{bmatrix}\right)^*A\\
&=\left(\begin{bmatrix}1&0&1\\0&-1&1\\1&-1&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\0\end{bmatrix}\right)^*\begin{bmatrix}1&0&1\\0&-1&1\\1&-1&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&-1&1\\1&-1&0\end{bmatrix}^{-1}\\
&=\left(\begin{bmatrix}1&0&1\\0&-1&1\\1&-1&0\end{bmatrix}\begin{bmatrix}y_1\\y_2\\0\end{bmatrix}\right)^*\begin{bmatrix}1&0&1\\0&-1&1\\1&-1&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1/2&-1/2&1/2\\1/2&-1/2&-1/2\\1/2&1/2&-1/2\end{bmatrix}\\
&=\begin{bmatrix}\frac{y_1^*-y_2^*}{2}&\frac{y_1^*-y_2^*}{2}&-\frac{y_1^*-y_2^*}{2}\end{bmatrix}\\
&\neq 0,\end{aligned}$$

$\mathscr{N}(A)$ is not orthogonal to $\mathscr{R}(A)$, which means that $\mathscr{R}(A)\neq\mathscr{R}(A^*)$.

Remark 5.2. In each case, for a given A, to guarantee the validity of (5.1), we only need to adjust M, e.g., we can choose α such that $M=\mathscr{H}(A)+\alpha I$, $\alpha>0$, satisfying (5.1). Then, there are many ways to construct Hermitian convergent splitting $M=F-G$, e.g., we can choose F as a diagonal or a tridiagonal matrix, with the purpose that the inverse of F is easier to construct than that of M.

Remark 5.3. We only focus on the convergence proof for the stationary case. It is natural to ask if we can estimate the convergence rate or a condition number. These are more complicated issues and constitute topics of our future research.

Acknowledgments

The author would like to thank the referees for their very careful readings of the original manuscript and many valuable comments and corrections.

References

[1] Z.-Z. Bai, D.-R. Wang, The monotone convergence of the two-stage iterative method for solving large sparse systems of linear equations, Appl. Math. Lett. 10 (1) (1997) 113–117.

[2] Z.-Z. Bai, The convergence of the two-stage iterative method for Hermitian positive definite linear systems, Appl. Math. Lett. 11 (2) (1998) 1–5.
[3] Z.-Z. Bai, S.-X. Qiu, Splitting-MINRES methods for linear systems with the coefficient matrix with a dominant indefinite symmetric part, Math. Numer. Sin. 24 (1) (2002) 113–128 (in Chinese).
[4] Z.-Z. Bai, G.H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM J. Matrix Anal. Appl. 24 (2003) 603–626.
[5] Z.-Z. Bai, C.-L. Wang, Convergence theorems for parallel multisplitting two-stage iterative methods for midly nonlinear systems, Linear Algebra Appl. 362 (2003) 237–250.
[6] A. Ben-Isreal, T.N.E. Greville, Genaralized Inverses, Theorey and Applications, Wiley, New York, 1974.
[7] A. Berman, R.-J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Academic Press, New York, 1979.
[8] Z.-H. Cao, Convergence of two-stage iterative methods for the solution of linear systems, Math. Numer. Sin. 17 (1995) 98–109 (in Chinese).
[9] Z.-H. Cao, On convergence of nested stationary iterative methods, Linear Algebra Appl. 221 (1995) 159–170.
[10] Z.-H. Cao, H.-B. Wu, Z.-Y. Liu, Two-stage iterative methods for consistent Hermitian positive semidefinite systems, Linear Algebra Appl. 294 (1999) 217–238.
[11] Z.-H. Cao, Convergence of block two-stage iterative methods for symmetric positive definite systems, Numer. Math. 90 (2001) 47–63.
[12] A. Frommer, D.B. Szyld, H-splitting and two-stage iterative methods, Numer. Math. 63 (1992) 345–356.
[13] A. Frommer, D.B. Szyld, Asynchronous two-stage iterative methods, Numer. Math. 69 (1994) 141–153.
[14] G.H. Golub, M.L. Overton, The convergence of inexact Chebyshev and Richardson iterative methods for solving linear systems, Numer. Math. 53 (1988) 571–593.
[15] V. Migállon, J. Penadés, D.B. Szyld, Block two-stage methods for singular systems and Markov chains, Numer. Linear Algebra Appl. 3 (1996) 413–426.
[16] V. Migállon, J. Penadés, Convergence of two-stage iterative methods for Hermitian positive definite matrices, Appl. Math. Lett. 10 (1997) 79–83.
[17] N.K. Nichols, On the convergence of two-stage iterative processes for solving linear equations, SIAM J. Numer. Anal. 10 (1973) 460–469.
[18] J.M. Ortega, Numerical Analysis – A Second Course, Academic Press, New York, 1972.
[19] Y.-Z. Song, L. Wang, Semiconvergence of *P*-regular splittings for singular linear systems, Preprint.
[20] C.-L. Wang, Z.-Z. Bai, Sufficient conditions for the convergent splittings of non-Hermitian positive definite matrices, Linear Algebra Appl. 330 (2001) 215–218.
[21] J.-Y. Yuan, The Ostrowski-Reich theorem for SOR iterations: extensions to the rank deficient case, Linear Algebra Appl. 315 (2000) 189–196.